

Divisibility properties in ultrapowers of commutative rings

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Abstract. All rings considered are commutative with identity. We study the preservation of certain properties in the passage from a ring R to the ultrapower R^* relative to a free ultrafilter on the set of all positive integers. Our main result is that if R is a locally pseudo-valuation domain (LPVD) of finite character (for instance, a semi-quasilocal LPVD), then R^* is also an LPVD. In the same vein, it is shown that the classes of pseudo-valuation domains and pseudo-valuation rings are each stable under the passage from R to R^*. An example is given of a divided domain R such that the domain R^* is not divided. A divisibility condition is found which characterizes the divided (respectively, quasilocal) rings R such that R^* is a divided (respectively, treed) ring.

Keywords. Ultrapower, linearly ordered, divided prime, divided ring, pseudo-valuation domain, ϕ-ring.

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1 Introduction

All rings considered in this note are commutative with identity. Our interest here is in the preservation of certain properties in the passage from a ring R to the ultrapower R^* relative to a free ultrafilter $\mathcal{U}$ on a denumerable index set I. For convenience, we identify I with the set $\mathbb{N}$ of all positive integers. (The interested reader is invited to check that our methods extend to the case in which $\mathcal{U}$ is any countably incomplete ultrafilter on an infinite index set I; and that many of our results carry over to ultraproducts.) By definition, R^* is the factor ring of $\prod_I R$ modulo the ideal $\{(a_i)_{i\in I} \mid Z(a_i) \in \mathcal{U}\}$, where $Z(a_i)$ denotes the set of coordinates i where $a_i = 0$. By an abuse of notation, we will also denote the elements of R^* by $(a_i)_{i\in I}$. It will be clear from the context whether we are working in the product or the ultrapower.

For some time, there has been considerable interest in the transfer of ring-theoretic properties between R and R^*; see, for instance, [19], [20], [21], as well as [16, pp. 179–180] for a brief introduction to ultrafilters and ultraproducts. Among the assembled lore is the fact that if R is an integral domain with quotient field K, then R^* is an integral domain with quotient field K^*. More significantly, if a ring R is semi-quasilocal with exactly n maximal ideals, then R^* is also semi-quasilocal with exactly n maximal ideals [20]. In particular, if (R, M) is a quasilocal ring, then R^* is also quasilocal, with unique maximal ideal M^*. Much of our motivation comes from the result [15] that if R is a Prüfer domain, then R^* is also a Prüfer domain. Hence, if R is a valuation

Note 1 replaced colon by semicolon

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domain, then so is R^*. In the same spirit, our main result, Corollary 3.3, establishes that if R is a locally pseudo-valuation domain (LPVD, in the sense of [12]) of finite character (for instance, a semi-quasilocal LPVD), then R^* is also an LPVD. Section 3 is devoted to a proof of Corollary 3.3, together with the supporting technical results on ultrafilters. Section 2 deals with easier material, primarily for certain quasilocal rings, suggested by the Prüfer $\leftrightarrow$ LPVD interplay.

The "interplay" that was just mentioned refers to the fact that LPVDs are, perhaps, the best behaved members in the class of locally divided integral domains (in the sense of [10]). This class is particularly interesting for several reasons: it contains all Prüfer domains as well as many integral domains that are not necessarily integrally closed; and the "locally divided integral domain" concept figures in several characterizations of Prüfer domains. Generalizations to the context of rings possibly with nontrivial zero-divisors have led to concepts such as divided rings [3], locally divided rings [7], and pseudo-valuation rings ([14], [6]). Among the results in Section 2, we establish in Corollary 2.8 and Proposition 2.9 that the classes of pseudo-valuation domains and pseudo-valuation rings are each stable under the passage from R to R^*. However, Example 2.4 shows that the class of divided domains does not exhibit similar stability. Section 2 also contains sharper facts, such as Example 2.5, involving the concept of a treed ring. (Recall that a ring R is called *treed* in case no maximal ideal of R can contain incomparable prime ideals of R; thus, a ring R is quasilocal and treed if and only if $\mathrm{Spec}(R)$, the set of all prime ideals of R, is linearly ordered with respect to inclusion.) The study of the various classes of divided rings in Section 2 is aided by a divisibility condition established in Proposition 2.3 as a characterization of rings R such that R^* is divided. This result is paired naturally with our first result, Proposition 2.1 which, in the context of ultrapowers, permits a permutation in the quantifications in a characterization of rings R such that $\mathrm{Spec}(R)$ is linearly ordered [2, Theorem 0].

Our reasoning with ultrapowers often depends on a number of facts that are used without further mention. For instance, consider elements $x = (x_i), y = (y_i)$ of the ultrapower R^*. Then $x^k = 0$ (for some $k \in \mathbb{N}$) if and only if $\{i \in I \mid x_i^k = 0\} \in \mathcal{U}$. Similarly, $x|y$ if and only if $\{i \in I \mid x_i|y_i\} \in \mathcal{U}$. Also, note that if P is an ideal of a ring R, then $P^* := \{(a_i) \mid \{i \in I \mid a_i \in P\} \in \mathcal{U}\}$ is an ideal of R^*; and $P^* \in \mathrm{Spec}(R^*)$ if and only if $P \in \mathrm{Spec}(R)$. Viewing R as canonically embedded in R^* via the diagonal map, we see easily that $P^* \cap R = P$. Furthermore, there are canonical isomorphisms $(R/P)^* \cong R^*/P^*$ and (if P is a prime ideal) $(R_P)^* \cong R^*_{P^*}$.

In addition to the notation and conventions mentioned above, we use dim(ension) to refer to Krull dimension; and, for a ring R, we use $\mathrm{Max}(R)$ to denote the set of maximal ideals of R, $\mathrm{Min}(R)$ to denote the set of minimal ideals of R, $\mathrm{Nil}(R)$ to denote the set of nilpotent elements of R, and $\mathrm{Rad}(J)$ to denote the nilradical of an ideal J of R. Any unexplained material is standard, as in [13].

2 Comparability properties in Spec(R^*)

Recall from [2, Theorem 0] that for any ring R, $\mathrm{Spec}(R)$ is linearly ordered (with respect to inclusion) if and only if, for any $a, b \in R$, there exists $n = n(a, b) \in \mathbb{N}$

Note 2 title: lowercase

such that either $a|b^n$ or $b|a^n$. This criterion is sharpened in the following result for ultrapowers. One consequence of Proposition 2.1 is the fact that if R is a ring such that $\mathrm{Spec}(R^*)$ is linearly ordered, then $\mathrm{Spec}(R)$ is also linearly ordered; a more elementary proof of this fact follows since $P^* \cap R = P$ for each $P \in \mathrm{Spec}(R)$.

Proposition 2.1. *Let R be a ring. Then* $\mathrm{Spec}(R^*)$ *is linearly ordered if and only if there exists $n \in \mathbb{N}$ such that for all $a, b \in R$, either $a|b^n$ or $b|a^n$.*

Proof. We first prove the contrapositive of the "only if" assertion. Suppose, then, that there does not exist $n \in \mathbb{N}$ such that for all $a, b \in R$, one has that either $a|b^n$ or $b|a^n$. Thus, for each $n \in \mathbb{N}$, there exist elements $a_n, b_n \in R$ such that $a_n \nmid b_n^n$ and $b_n \nmid a_n^n$. Define $\alpha, \beta \in R^*$ by $\alpha := (a_n)_{n\in\mathbb{N}}$ and $\beta := (b_n)_{n\in\mathbb{N}}$. We claim that for all $n \in \mathbb{N}$, $\alpha \nmid \beta^n$ and $\beta \nmid \alpha^n$. Given the claim, one sees via the criterion in [2, Theorem 0] that, as desired, $\mathrm{Spec}(R^*)$ is not linearly ordered.

Suppose that the above claim fails. Then, without loss of generality, $\alpha|\beta^k$ for some $k \in \mathbb{N}$. It follows, by a fact recalled in the Introduction, that $\{n \in \mathbb{N} \mid a_n|b_n^k\} \in \mathcal{U}$. However, for all $n \geq k$, we know that $a_n \nmid b_n^k$. Hence, $\{n \in \mathbb{N} \mid a_n|b_n^k\} \subseteq \{1, 2, \ldots, k-1\}$. This is a contradiction, since a finite set cannot be a member of a free ultrafilter. This establishes the claim and completes the proof of the "only if" assertion.

We next turn to the "if" assertion. Suppose, then, that there exists $k \in \mathbb{N}$ such that for $a, b \in R$, either $a|b^k$ or $b|a^k$. By applying the above criterion from [2, Theorem 0], our task is translated to showing that if $\alpha := (a_n)_{n\in\mathbb{N}}$ and $\beta := (b_n)_{n\in\mathbb{N}}$, then there exists $n \in \mathbb{N}$ such that either $\alpha|\beta^n$ or $\beta|\alpha^n$. It suffices to show that either $\alpha|\beta^k$ or $\beta|\alpha^k$. Putting $V := \{i \in \mathbb{N} \mid a_i|b_i^k\}$ and $W := \{i \in \mathbb{N} \mid b_i|a_i^k\}$, we see from a fact recalled in the Introduction that an equivalent task is to show that either $V \in \mathcal{U}$ or $W \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it is enough to show that $V \cup W = \mathbb{N}$. This equality is, however, ensured by the hypothesis of the "if" assertion, to complete the proof. □

Recall that a local (Noetherian) integral domain (R, M) is called *analytically unramified* (resp., *analytically irreducible*) if its completion with respect to the filtration given by the powers M^n is a reduced ring (resp., an integral domain). Our next result, which re-encounters the criterion from Proposition 2.1, is also interesting in that neither its hypothesis nor its conclusion mentions an ultrapower.

Corollary 2.2. *Let R be an analytically unramified one-dimensional local integral domain. Then R is analytically irreducible if and only if there exists $n \in \mathbb{N}$ such for all $a, b \in R$, either $a|b^n$ or $b|a^n$.*

Proof. The integral closure of R (in its quotient field) is a finitely generated R-module [22]. It therefore follows from well-known results (cf. [18, (43.20), (32.2)], [8, Proposition III.5.2]) that R is analytically irreducible if and only if the integral closure of R is local. By [21, Theorem 6.3], the latter condition is equivalent to requiring that $\mathrm{Spec}(R^*)$ is linearly ordered. Accordingly, an application of Proposition 2.1 completes the proof. □

Note 3 replaced 'well known' by 'well-known'

A ring R is called *divided* if, for each $P \in \mathrm{Spec}(R)$ and $a \in R$, either $a \in P$ or $P \subseteq Ra$. Recall from [3, Proposition 2] that a ring R is divided if and only if, for any elements $a, b \in R$, either $a|b$ or there exists $n = n(a,b) \in \mathbb{N}$ such that $b|a^n$. It therefore follows that one consequence of Proposition 2.3 is the fact that if R is a ring such that R^* is divided, then R is also divided. The proof of Proposition 2.3 is similar to the proof of Proposition 2.1 and is hence omitted.

Proposition 2.3. *Let R be a divided ring. Then R^* is divided if and only if there exists $n \in \mathbb{N}$ such that for all $a, b \in R$, either $a|b$ or $b|a^n$.*

We next construct an example to show that a divided ring R need not satisfy the divisibility condition in the statement of Proposition 2.3

Example 2.4. There exists a divided ring R such that R^* is not divided. Our construction uses an infinite strictly ascending chain of fields $\mathbb{Q} \subset F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$, with $F := \cup F_n$. Fix a prime integer p and let

Note 4 replace cup by bigcup (also in other (similar) places)?

$$R := \mathbb{Z}_{(p)} + F_1X + F_2X^2 + \cdots + F_nX^n + \cdots \subseteq F[[X]].$$

To show that R is divided, it will be convenient to first show that the ring

$$T := \mathbb{Q} + F_1X + F_2X^2 + \cdots + F_nX^n + \cdots \subseteq F[[X]]$$

is divided. For this, it suffices, by [3, Proposition 2], to show that if $a = \sum_{i=n}^{\infty} a_iX^i$ and $b = \sum_{j=m}^{\infty} b_jX^j$ are nonzero nonunits of T, with $1 \leq n \leq m$, then there exists a positive integer ν such that $a^\nu/b \in T$. Choose ν so that $\nu n > 2m$. Then $p := \nu n - m > m$ and we easily see by the usual process of long division that when a^ν $(= (a_n)^\nu X^{n\nu} +$ higher degree terms) is divided by b, the quotient in $F[[X]]$ actually lies in T. In other words, b divides a^ν in T, and so T is a divided domain. Next, consider the maximal ideal $M := F_1X + F_2X^2 + \cdots + F_nX^n + \cdots$ of T. Observe that $T/M \cong \mathbb{Q}$. Then, since R is the pullback $T \times_{T/M} \mathbb{Z}_{(p)}$ with both T and $\mathbb{Z}_{(p)}$ being divided domains, it follows from [11, Corollary 2.6] that R is a divided domain, and hence a divided ring, as asserted.

Moreover, we claim that R^* is not divided. To prove this claim, pick $d_n \in F_n \setminus F_{n-1}$ for each $n \in \mathbb{N}$. Set $a_n := X$ and $b_n := d_nX^n$. Then for each n, $a_n \nmid b_n$ and $b_n \nmid a_n^n$. Define $\alpha, \beta \in R^*$ via $\alpha := (a_n)_{n \in \mathbb{N}}$ and $\beta := (b_n)_{n \in \mathbb{N}}$. Then $\alpha \nmid \beta$ and $\beta \nmid \alpha^n$ for all $n \in \mathbb{N}$. So, by [3, Proposition 2], R^* is not divided, thus proving the above claim.

Let R be the divided ring in the above example. Although R^* is not divided, note that R^* *is* treed (in contrast to the situation in Example 2.5 below). To see this, observe that for all $a, b \in R$, either $a|b^2$ or $b|a^2$, and apply Proposition 2.1.

Next, we present a family of examples of divided rings whose ultrapowers are not treed.

Example 2.5. There exists a divided ring R such that R^* is not treed. Indeed, consider any analytically unramified one-dimensional local integral domain R. Trivially, R is a divided ring. Moreover, as recalled in the Introduction, R^* inherits the “quasilocal” condition from R. Therefore, by also arranging that R is not analytically irreducible (for instance, taking R as in the examples in [9, pp. 54–55]), we see from Corollary 2.2 and Proposition 2.1 that R^* is not treed.

Our next result shows that the property involving a uniform bound that was mentioned in the hypothesis of Proposition 2.3 ascends and descends in the context of a certain pullback, namely, the "$\mathrm{Spec}(R) = \mathrm{Spec}(T)$" context of [1]. First, for $n \in \mathbb{N}$, it is convenient to say that a ring R has *property* $*_n$ if, for all elements $a, b \in R$, either $a|b$ or $b|a^n$. It is clear that if a ring R satisfies $*_n$ for some $n \in \mathbb{N}$, then R satisfies $*_m$ for all $m \geq n$.

Proposition 2.6. *Let $R \subset T$ be quasilocal rings with common maximal ideal M. Then:*

(a) *If R satisfies $*_n$, then T satisfies $*_n$.*

(b) *If T satisfies $*_1$, then R satisfies $*_2$.*

(c) *If T satisfies $*_n$ for some $n \geq 2$, then R satisfies $*_n$.*

Proof. (a) The assertion is clear because R and T have the same set of nonunits.

(b) Suppose that T satisfies $*_1$, and let $a, b \in R$. We need to show that either $a|b$ in R or $b|a^2$ in R. Without loss of generality, neither a nor b is a unit; thus, $a, b \in M$. Suppose that $a \nmid b$ in R. We show that $b|a^2$ in R. There are two cases.

In the first case, $a|b$ in T. Then $b = ax$ for some $x \in T \setminus R$. Since $M \subset R$, we conclude that x is a unit of T. Then $a = x^{-1}b$ (in T), and so $a^2 = (ax^{-1})b \in Rb$, as $ax^{-1} \in M \subset R$. In particular, $b|a^2$ in R.

In the remaining case, $a \nmid b$ in T. Then, by hypothesis, $b|a$ in T. Write $by = a$, with $y \in T$. It follows that $a^2 = b(ay)$ and so, since $ay \in M \subset R$, we have $b|a^2$ in R, as desired.

(c) Suppose that T satisfies $*_n$ for some $n \geq 2$. As in the proof of (b), we must show that if $a, b \in M$, then either $a|b$ in R or $b|a^n$ in R. Suppose that $a \nmid b$ in R. We show that $b|a^n$ in R. If $a|b$ in T, we can argue as in the proof of (b) to show that $b|a^2$ in R, whence $b|a^n$ in R. Thus, without loss of generality, $a \nmid b$ in T. Then, by hypothesis, $b|a^n$ in T. Write $a^n = bx$, with $x \in T$. If $x \in R$, we are done, and so we may assume that x is a unit of T. Since $n \geq 2$, we have $b = x^{-1}a^n = (x^{-1}a^{n-1})a$, with $(x^{-1}a^{n-1}) \in M \subset R$. Thus, in the case to which we have reduced, it follows that $a|b$ (in T), a contradiction. The proof is complete. □

Proposition 2.7. *Let T be an overring of an integral domain R. If Spec$(R) =$ Spec(T) (as sets), then Spec$(R^*) =$ Spec(T^*).*

Proof. Without loss of generality $R \neq T$. Then, by [1, Lemma 3.2], R and T are quasilocal, with the same maximal ideal, say M. By a fact recalled in the Introduction, it follows that R^* and T^* are each quasilocal with unique maximal ideal M^*. Therefore, by [1, Proposition 3.8], $\mathrm{Spec}(R^*) = \mathrm{Spec}(T^*)$. □

Recall from [14] that a quasilocal domain (R, M) is called a *pseudo-valuation domain* (PVD) if R has a (uniquely determined) valuation overring V such that V has maximal ideal M; equivalently, such that $\mathrm{Spec}(R) = \mathrm{Spec}(V)$ (as sets).

Corollary 2.8. *If R is a PVD, then R^* is a PVD.*

Proof. Since the ultrapower of a valuation domain is also a valuation domain, the result follows by combining Proposition 2.7 and the second of the above characterizations of PVDs. □

The preceding result generalizes to arbitrary (commutative) rings. In the process of proving this (see Proposition 2.9 below), we make contact with the following interesting class of divided rings. Recall from [6] that a ring R is called a *pseudo-valuation ring* (PVR) if Pa and Rb are comparable (with respect to inclusion) for all $P \in \operatorname{Spec}(R)$ and $a, b \in R$. An integral domain is a PVR if and only if it is a PVD. Any PVR is a divided, hence quasilocal, ring. It was shown in [6] that a quasilocal ring (R, M) is a PVR if and only if, for all elements $a, b \in R$, either $a|b$ or $b|am$ for each $m \in M$.

Proposition 2.9. *If R is a PVR, then R^* is a PVR.*

Proof. By the above remarks, R is quasilocal, say with maximal ideal M. Therefore, R^* is quasilocal, with maximal ideal M^*. Consider arbitrary elements $\alpha = (a_n)_{n\in\mathbb{N}}, \beta = (b_n)_{n\in\mathbb{N}} \in R^*$ and $\mu = (m_n)_{n\in\mathbb{N}} \in M^*$. Without loss of generality, the coordinates m_n may be chosen so that $m_n \in M$ for each $n \in \mathbb{N}$. Put $V := \{i \in \mathbb{N} \mid a_i|b_i\}$ and $W := \{i \in \mathbb{N} \mid b_i|a_i m_i\}$. Since R is a PVR, the last-mentioned characterization of PVRs yields that $V \cup W = \mathbb{N}$. As $\mathcal{U}$ is an ultrafilter, it follows that either $V \in \mathcal{U}$ or $W \in \mathcal{U}$. In the first (resp., second) case, $\alpha|\beta$ (resp., $\beta|\alpha\mu$). Thus, either $\alpha|\beta$ or $\beta|\alpha\mu$ for each $\mu \in M^*$. In other words, R^* is a PVR. □

Recently, much attention has been paid to a certain class $\mathcal{C}$ of divided rings that contains the class of PVRs. We next recall the definition of $\mathcal{C}$ and show that, unlike the classes of Prüfer domains, valuation domains, PVDs and PVRs, $\mathcal{C}$ is *not* stable under the passage from R to R^*.

Recall that a prime ideal P of a ring R is said to be *divided (in R)* if P is comparable (with respect to inclusion) to Rb for each $b \in R$. A ring R is called a *Φ-pseudo-valuation ring* (Φ-PVR) if $\operatorname{Nil}(R)$ is a divided prime ideal of R and, for all $a, b \in R \setminus \operatorname{Nil}(R)$, either $a|b$ or $b|an$ for all nonunits n of R. The following particularly useful characterization of the Φ-PVR concept appears in [5]. A ring R is a Φ-PVR if and only if $\operatorname{Nil}(R)$ is a divided prime ideal of R and $R/\operatorname{Nil}(R)$ is a PVD.

Example 2.10. There exists a Φ-PVR R such that R^* is not a Φ-PVR. It can be further arranged that $\operatorname{Nil}(R)$ is a prime ideal of R but $\operatorname{Nil}(R^*)$ is not a prime ideal of R^*. For a construction of such, begin with any field K and set

$$R := K[X_1, X_2, \dots, X_n, \dots]/(\{X_n^n, X_i X_j \mid n, i, j \in \mathbb{N}, i \neq j\}).$$

Notice that the images of the X_i's are nilpotent and generate the unique maximal ideal, say M, of R. In particular, $\operatorname{Nil}(R) = M \in \operatorname{Spec}(R)$. It is then trivial via the above criterion from [5] that R is a Φ-PVR. To show that R^* is not a Φ-PVR, we produce elements $\alpha \in \operatorname{Nil}(R^*)$, $\beta \in R^* \setminus \operatorname{Nil}(R^*)$ such that $\beta \nmid \alpha$. Observe that the elements $\alpha := (X_2, X_3, X_2, X_3, \dots)$ and $\beta := (X_1, X_2, X_3, \dots, X_n, \dots)$ have the asserted properties. Thus, $\operatorname{Nil}(R^*)$ is not a divided prime ideal of R^*, and so R^* is not a Φ-PVR.

It remains to verify that $\mathrm{Nil}(R^*)$ is not a prime ideal of R^*. Consider the element $\gamma := (X_2, X_3, \ldots, X_n, \ldots) \in R^*$. Evidently, $\beta\gamma = 0$. We noted above that β is not nilpotent; and in the same way, one checks that γ is not nilpotent. The verification is complete.

In view of the unexpected behavior of $\mathrm{Nil}(R^*)$ in the preceding example, we devote the final two results of this section to additional scrutiny of related behavior. We begin by analyzing the behavior of "Rad" in the passage from R to R^*.

Remark 2.11. Let J be an ideal of a ring R. Then $\mathrm{Rad}(J^*) \subseteq \mathrm{Rad}(J)^*$, with equality if and only if there exists $n \in \mathbb{N}$ such that $a^n \in J$ for all $a \in \mathrm{Rad}(J)$. The proof is similar to that of Proposition 2.1; see [19, Proposition 2.28].

Note 5 replaced colon by semicolon

Example 2.12. There exists a PVR, R, such that $\mathrm{Nil}(R)$ is a prime ideal of R and $\mathrm{Nil}(R^*)$ is a prime ideal of R^*, but $\mathrm{Nil}(R^*) \neq \mathrm{Nil}(R)^*$. (By taking $J := 0$ in the preceding remark, one trivially has that $\mathrm{Nil}(S^*) \subseteq \mathrm{Nil}(S)^*$ for any ring S.) For a construction, consider any rank one non-discrete valuation domain (D, M). Choose any nonzero element $d \in M$; then $\mathrm{Rad}(Dd) = M$. Then, of course, [6, Corollary 3] ensures that $R := D/Dd$ is a PVR, and so $\mathrm{Nil}(R)$ is a prime (in fact, the unique maximal) ideal of R. Moreover, by Proposition 2.9, R^* is a PVR, and so $\mathrm{Nil}(R^*)$ is a prime ideal of R^*.

It remains to verify that $\mathrm{Nil}(R^*) \neq \mathrm{Nil}(R)^*$. Deny. Then, by the criterion in Remark 2.11, there exists $n \in \mathbb{N}$ such that $a^n = 0$ for all $a \in \mathrm{Rad}(0) = \mathrm{Nil}(R)$. Therefore, for all $m \in M$, we have $m^n \in Dd$. Letting v denote any (real-valued) valuation associated to D, we infer the existence of $d_1 \in D$ such that

$$nv(m) = v(m^n) = v(d_1 d) = v(d_1) + v(d) \geq v(d),$$

whence $v(m) \geq \frac{v(d)}{n}$, an absurdity since the non-discreteness of D guarantees the existence of elements $m \in M$ with arbitrarily small (positive) v-value. This (desired) contradiction completes the verification.

3 Ultrapowers of LPVDs

For the remainder of the paper, R will denote an integral domain. Recall from [12] that R is called a *locally pseudo-valuation domain* (LPVD) if R_P is a PVD for all $P \in \mathrm{Spec}(R)$ (equivalently, for all $P \in \mathrm{Max}(R)$). We will show that with an extra assumption on R, the property of being an LPVD is inherited by R^*. First, we need to describe the maximal ideals of R^*. In [23], a description of some of the maximal ideals of the product $T := \prod_I R$ is given (an equivalent formulation is given in [20]). Furthermore, it is shown in [23] that with an additional hypothesis, these are all the maximal ideals of T. We next present that description here and then pass to considerations involving the factor ring R^*.

Let $\mathcal{J}$ denote the set of all functions from the index set I to the set of finite subsets of $\mathrm{Max}(R)$. For $\sigma, \rho \in \mathcal{J}$, we say $\rho \leq \sigma$ (ρ is called a *subfunction of* σ) if $\rho(i) \subseteq \sigma(i)$ for all $i \in I$. For $\rho \leq \sigma \in \mathcal{I}$, we define $\sigma \setminus \rho$ to be the function given by $\sigma(i) \setminus \rho(i)$ for each $i \in I$. Also, we define the functions $\sigma \vee \rho$ (resp., $\sigma \wedge \rho$) via $(\sigma \vee \rho)(i) := \sigma(i) \cup \rho(i)$ (resp., $(\sigma \wedge \rho)(i) := \sigma(i) \cap \rho(i)$) for each $i \in I$. Finally, the *blank function* Φ is defined by $\Phi(i) := \emptyset$ for each $i \in I$. Now, consider a fixed element $\sigma \in \mathcal{I}$. The set of subfunctions of σ forms a Boolean algebra with σ as 1, Φ as 0, and the complement of $\rho \leq \sigma$ is $\rho' = \sigma \setminus \rho$. Therefore, it makes sense to talk about ultrafilters on the set of subfunctions of σ. In particular, by an *ultrafilter on* σ, we mean a collection of functions $\mathcal{F} \subseteq \{\rho \mid \rho \leq \sigma\}$ such that:

- $\sigma \in \mathcal{F}$ and $\Phi \notin \mathcal{F}$;
- If $\rho \in \mathcal{F}$ and $\rho \leq \tau$, then $\tau \in \mathcal{F}$;
- If $\rho, \tau \in \mathcal{F}$, then $\rho \wedge \tau \in \mathcal{F}$;
- If ρ is an element of the Boolean algebra of subfunctions of σ, then either $\rho \in \mathcal{F}$ or $\rho' \in \mathcal{F}$.

If $\mathcal{F}$ is an ultrafilter on σ, then by a standard argument, one can show that if $\rho \vee \tau \in \mathcal{F}$, then either $\rho \in \mathcal{F}$ or $\tau \in \mathcal{F}$.

For $a = (a_i)_{i \in I} \in T$, we can obtain an interesting subfunction of σ by defining $\sigma_a(i) := \{P \in \sigma(i) \mid a_i \in P\}$. Now, consider any ultrafilter $\mathcal{F}$ on σ. Set $(\mathcal{F}) := \{a \in T \mid \sigma_a \in \mathcal{F}\}$. As shown in [17] or [23], $(\mathcal{F})$ is a maximal ideal of T. Moreover, if each nonzero element of R is contained in only finitely many maximal ideals of R (such rings R are said to have *finite character*), then these $(\mathcal{F})$'s are all the maximal ideals of T [23, Theorem 1.2].

It is well known that for a (commutative) ring S, each $P \in \mathrm{Spec}(S)$ induces an ultrafilter $\mathcal{U}_P$ on the Boolean algebra of idempotents of S via: $e \in \mathcal{U}_P$ if and only if $1 - e \in P$. Furthermore, it is easy to see that if $P \subseteq Q$ are elements of $\mathrm{Spec}(S)$, then $\mathcal{U}_P \subseteq \mathcal{U}_Q$; hence, since $\mathcal{U}_P$ and $\mathcal{U}_Q$ are maximal filters, we have $\mathcal{U}_P = \mathcal{U}_Q$. Now if (as above) $T := \prod_I R$, where R is an integral domain, then the Boolean algebra of idempotents of T is isomorphic to the Boolean algebra of subsets of I. Thus, each prime ideal P of T determines an ultrafilter $\mathcal{U}_P$ on (the Boolean algebra of subsets of) I. In particular, $\mathcal{U}_P := \{A \subseteq I \mid 1 - e_A \in P\}$, where e_A denotes the characteristic function on A.

Conversely, given an ultrafilter $\mathcal{U}$ on (the Boolean algebra of subsets of) I, one can construct an ideal $P_{\mathcal{U}} \subset T$ by setting $P_{\mathcal{U}} := (\{1 - e_A \mid A \in \mathcal{U}\})$. Observe that if $(a_i) \in T$, then $(a_i)(1 - e_A) = (a_i)$, where $A := Z(a_i)$. Thus $P_{\mathcal{U}}$ is the ideal of relations that defines the ultrapower R^* and so $T/P_{\mathcal{U}} = R^*$. Therefore, if R is an integral domain, $P_{\mathcal{U}}$ is a prime ideal of T. Furthermore, one sees that $\mathcal{U}_{P_{\mathcal{U}}} = \mathcal{U}$ and $P_{\mathcal{U}_P} \subseteq P$. Hence the assignment $\mathcal{U} \mapsto P_{\mathcal{U}}$ defines a bijection between the set of ultrafilters on I and $\mathrm{Min}(T)$.

In addition, we claim that each $Q \in \mathrm{Spec}(T)$ contains a unique minimal prime ideal P. Suppose the claim is false, and take $Q \in \mathrm{Spec}(T)$ such that Q contains two distinct minimal prime ideals P_1 and P_2. Since each minimal prime ideal of T is generated by idempotents, there exists an idempotent $e \in P_1 \setminus P_2$. Therefore $1 - e \in P_2$, whence $1 = e + (1 - e) \in Q$, a contradiction, thus proving the claim.

Furthermore, suppose $Q \in \mathrm{Spec}(T)$, and let P denote the unique minimal prime ideal of T that is contained in Q. Then, since the only idempotents in a local ring are 0 and 1, it follows that $PT_Q = 0$. Hence, there are canonical isomorphisms $T_Q \cong T_Q/PT_Q \cong (T/P)_{Q/P}$.

Now, we return to ultrapowers of an integral domain R with respect to an ultrafilter $\mathcal{U}$ on I. As noted above, $R^* = T/P_{\mathcal{U}}$. Therefore, to examine the localization of R^* at an arbitrary maximal ideal, it suffices, by the preceding remarks, to consider the localization of T at a suitable maximal ideal. First, we need the following technical lemma.

Lemma 3.1. *Let $\mathcal{F}$ be an ultrafilter on $\sigma \in \mathcal{I}$ and let $\tau \in \mathcal{F}$. Let $M := (\mathcal{F})$ be the maximal ideal determined by $\mathcal{F}$ and put $\mathcal{U} := \mathcal{U}_M$. Then there exists $W \in \mathcal{U}$ such that $\tau(i) \neq \varnothing$ for all $i \in W$.*

Proof. Partition the set I into $V := \{i \in I \mid \tau(i) = \varnothing\}$ and $W := I \setminus V$. Observe that $\sigma_{e_V} \geq \tau \in \mathcal{F}$, whence $\sigma_{e_V} \in \mathcal{F}$. Thus, $e_V \in M = (\mathcal{F})$. Since $\mathcal{U} = \mathcal{U}_M$, it follows that $W \in \mathcal{U}$, and the result is proved. □

Before moving on to our result for LPVDs, we give some general definitions. Let R be a ring with $a, b \in R$ and let S be a multiplicative subset of R. We say that *a divides b with respect to S* if the image of a divides the image of b in the ring R_S. This is equivalent to saying that there exists $r \in R$ and $s \in S$ such that $ar = bs$. Note that if X is any finite subset of $\mathrm{Max}(R)$, then a divides b with respect to all $R \setminus P$ for all $P \in X$ if and only if a divides b with respect to $R \setminus \cup_{P \in X} P$.

Theorem 3.2. *Let R be an LPVD and let $(\mathcal{F})$ be the maximal ideal of T determined by an ultrafilter $\mathcal{F}$ on some σ. Then $T_{(\mathcal{F})}$ is a PVD.*

Proof. Let $M = (\mathcal{F})$. To show that T_M is a PVD, we must show that given any two elements $\alpha, \beta \in M$, either α divides β with respect to $T \setminus M$ or β divides αm with respect to $T \setminus M$ for all $m \in M$ (cf. [6]).

Let $\alpha = (a_i)_{i \in I}$ and $\beta = (b_i)_{i \in I}$ be elements of M. Thus the element $\sigma_\alpha \wedge \sigma_\beta$ is in $\mathcal{F}$. Define $\tau \leq \sigma_\alpha \wedge \sigma_\beta$ by $\tau(i) := \{P \in (\sigma_\alpha \wedge \sigma_\beta)(i) \mid a_i \text{ divides } b_i \text{ with respect to } R \setminus P\}$. Set $\rho := (\sigma_\alpha \wedge \sigma_\beta) \setminus \tau$.

Since $\mathcal{F}$ is an ultrafilter, either $\tau \in \mathcal{F}$ or $\rho \in \mathcal{F}$. First, assume the former. In this case, we claim that α divides β with respect to $T \setminus M$. To see this, note that by Lemma 3.1, there exists $W \in \mathcal{U}$ such that for all $i \in W$, $\tau(i) \neq \varnothing$. Furthermore, from the definition of τ, it follows that for each $i \in W$, there exists $r_i \in R$ and $s_i \in R \setminus \cup_{P \in \tau(i)} P$ such that $a_i r_i = b_i s_i$. For all $i \in I \setminus W$, let $r_i := s_i := 1$. Use this data to define two elements of R^*, namely, $r := (r_i)$ and $s := (s_i)$. It follows that $\alpha r = \beta s$. However, it is also clear from the definition that $s \notin M$, thus proving the claim.

In the remaining case, we can assume that $\rho \in \mathcal{F}$. Let $m := (m_i)_{i \in I} \in M$. It suffices to show that β divides αm with respect to $T \setminus M$. Since $\sigma_m \in \mathcal{F}$, we have $\sigma_m \wedge \rho \in \mathcal{F}$. Therefore, it again follows from Lemma 3.1 that there exists $W \in \mathcal{U}$ such that for all $i \in W$, $(\sigma_m \wedge \rho)(i) \neq \varnothing$. Also from the definition of ρ, a_i does not divide b_i with respect to $R \setminus P$ for any $P \in (\sigma_m \wedge \rho)(i)$. Therefore, since R is an LPVD, we have that for each $i \in W$, there exist $s_i \in R \setminus \cup_{P \in (\sigma_m \wedge \tau)(i)} P$ and $r_i \in R$ such that $m_i a_i s_i = b_i r_i$. For all other i, set $r_i := s_i := 1$. Thus, $s := (s_i) \in T \setminus M$, $r := (r_i) \in T$, and $m\alpha s = \beta r$. The last equation is the statement that β divides αm with respect to $T \setminus M$, as desired. □

Corollary 3.3. *Let R be an LPVD with finite character. Then any ultrapower R^* is also an LPVD.*

Proof. Since R has finite character, it follows from [23, Theorem 1.2] that each maximal ideal of R^* is the image of a maximal ideal of T of the form $(\mathcal{F})$. Thus, the result follows directly from Theorem 3.2. □

We close with two observations. First, the "locally divided" analogue of the preceding result is false. Indeed, Examples 2.4 and 2.5 each show that if R is a quasilocal locally divided integral domain (necessarily of finite character), in other words a divided domain, then R^* need not be locally divided. Second, we do not know whether Corollary 3.3 remains valid if one deletes the "finite character" hypothesis.

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